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Scattering operators on Fock space: II. Representations of the scattering operator generated by SL(2, $\mathbb{R}$)

W H Klink

Department of Physics and Astronomy, University of Iowa, Iowa City, IA 52242, USA

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Abstract. A representation for unitary scattering operators acting on a symmetric Fock space and invariant under an SO(N) internal symmetry group is constructed. A group of transformations commuting with SO(N) is seen to be isomorphic to SL(2, $\mathbb{R}$); the representations of SL(2, $\mathbb{R}$) acting on the Fock space are shown to come from the discrete series of representations of SL(2, $\mathbb{R}$). These representations are used to label the equivalent irreducible representations of SO(N), and the partial wave amplitudes of the scattering operators are shown to be matrix elements of the discrete series of representations of SL(2, $\mathbb{R}$). The example of isospin internal symmetry and the pion triplet is briefly discussed.

1. Introduction

To investigate strong interaction multiparticle phenomena from a scattering operator point of view, it is useful to begin with internal symmetries and see how a given representation of the internal symmetry group governs the structure of a general unitary invariant scattering operator. In this paper we continue the analysis begun in Klink (1985, hereafter referred to as I), where a compact internal symmetry group K, with a representation acting on a space V of dimension N, was associated with the group U(N).

For internal symmetries whose representation space V accommodates a multiplet of bosons, the relevant many-particle space is the symmetric Fock space, consisting of the direct sum of all n-fold symmetric tensor products of V. Unlike the group K where the n-fold tensor products are generally reducible, these n-fold symmetric tensor products are irreducible for U(N). In I it was shown that the symmetric Fock space $\mathbb{S}(V)$ is isomorphic to a Hilbert space of functions on U(N)/U($N-1$), induced by the identity representation of U($N-1$). A unitary invariant scattering operator was then constructed by having the kernel of the operator act only on the double cosets of U(N) with respect to U($N-1$) and K.

This gave a representation of unitary invariant partial wave amplitudes, $K_\chi(n, \eta; n', \eta')$ (I, equation (18)), where χ is the irreducible representation of K, $n(n')$ is the number of particles in the final (initial) state and $\eta(\eta')$ is a multiplicity label distinguishing between equivalent representations χ in the $n(n')$ particle subspace. Now the kernels $K_{DD'}$ of the scattering operators (labelled by double cosets D, D') form an infinite parameter group, which we denote by G^∞. G^∞ is, however, too big to be useful for dealing with the multiplicity problem. We would like to find a finite parameter subgroup G of G^∞ whose representations could be used to label the multiplicity η of equivalent representations of K.

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To see how to construct such a group, we consider in this paper $SO(N)$ internal symmetry groups, for which the decomposition of the representations of $U(N)$ appearing in the Fock space into $SO(N)$ irreducibles is well known. We will show that related to the $U(N-1)^{\backslash U(N)/}SO(N)$ double coset decomposition is a finite parameter subgroup of G^{∞} that is isomorphic to $SL(2, \mathbb{R})$, and further, that the discrete series of representations of $SL(2, \mathbb{R})$ completely label the multiplicity of $SO(N)$ representations in the Fock space.

Actually, the computations will not be carried out on Fock space or on the Hilbert space over the complex sphere introduced in I, but rather on a Gaussian Hilbert space. There spaces are introduced in § 3 and are used to construct operators that commute with $SO(N)$ and form a Lie algebra of $SL(2, \mathbb{R})$. In § 2 the ideas introduced in I are reviewed and the reduction of $U(N)$ to $SO(N)$ representations is given.

The discrete series of representations of $SL(2, \mathbb{R})$ can be used not only to label equivalent $SO(N)$ representations, but also to generate partial wave amplitudes of scattering operators, indexed by $SL(2, \mathbb{R})$ elements, that are unitary and invariant with respect to $SO(N)$. The unitarity of these partial wave amplitudes is a consequence of the unitarity of the discrete series of representations of $SL(2, \mathbb{R})$. Such partial wave amplitudes are, of course, not the most general unitary invariant partial wave amplitudes, since these arise from the infinite parameter group G^{∞}. They may, however, be useful in phenomenological analyses, such as arise in the example discussed in § 4, of an isospin internal symmetry $SU(2)$ and the pion triplet.

2. Connecting U(N) and SO(N) representations

As in I we begin with a compact group K and representation space V of dimension N. The symmetric Fock space $\mathbb{S}(V)$ is formed by taking the direct sum of n-fold symmetric tensor products of V:

$$\mathbb{S}(V) = \sum_{n=0}^{\infty} \oplus V_n \qquad V_n = (V \otimes \ldots \otimes V)_{\text{sym}}. \tag{1}$$

Now, as discussed in I, if V is a complex vector space, it also is a representation space for the fundamental representation of $U(N)$. For $U(N)$ the n-fold symmetric tensor space is irreducible and given (in Gel'fand notation) by $(n0 \ldots 0)$. However, for K the n-fold symmetric tensor space is generally reducible. In particular, if $K = SO(N)$, and acts on V as a subgroup of $U(N)$, the decomposition of the n-fold symmetric tensor space is given by

$$\begin{aligned} \overbrace{(n0 \ldots 0)}^{N} &= \overbrace{(n0 \ldots 0)}^{N/2} + \overbrace{(n-2, 0 \ldots 0)}^{N/2} + \ldots + \overbrace{(0 \ldots 0)}^{N/2} && n \text{ even} \\ \overbrace{(n0 \ldots 0)}^{N} &= \overbrace{(n0 \ldots 0)}^{\frac{1}{2}(N-1)} + \overbrace{(n-2, 0 \ldots 0)}^{\frac{1}{2}(N-1)} + \ldots + \overbrace{(10 \ldots 0)}^{\frac{1}{2}(N-1)} && n \text{ odd.} \end{aligned} \tag{2}$$

This shows that no irreducible representation of $SO(N)$ appears more than once in a given n-particle subspace of the full Fock space. Or, put differently, the multiplicity of $SO(N)$ irreducible representations in $\mathbb{S}(V)$ is completely specified by n, the number of particles.

For example, if K is the isospin group $SU(2)$ and V the representation space associated with a triplet of pions, then the group action in V can be chosen to be

SO(3). The representations of SO(3) contained in the U(3) representation $(n00)$ are given by

$$(n00) \to I = n, n-2, \ldots, 0 \qquad n \text{ even}$$

$$(n00) \to I = n, n-2, \ldots, 1 \qquad n \text{ odd}$$

where I is the isospin. This decomposition can be written in another way which will be important in the coming sections, namely a given I occurs in the $n = I, I+2, \ldots,$ subspaces of $\mathbb{S}(V)$; note that even (odd) values of n occur only with even (odd) values of I. We see that towers of particles are formed which will be associated with the discrete series of representations of $\mathrm{SL}(2, \mathbb{R})$.

More generally, if l denotes the $(l0\ldots0)$ representation of SO(N) (Barut and Raczka 1977), then the towers become $n = l, l+2, l+4, \ldots,$ again with no mixing between even and odd values. The $\mathrm{SL}(2, \mathbb{R})$ raising and lowering operators which commute with SO(N) have the property that they change n by ± 2, while leaving l unchanged.

In I $\mathbb{S}(V)$ was shown to be isomorphic to a Hilbert space defined on the homogeneous space $\mathrm{U}(N)/\mathrm{U}(N-1)$, and the scattering operator then acted as a unitary operator on the orbits of $\mathrm{U}(N)/\mathrm{U}(N-1)$ generated by the representation V of K. As discussed in the introduction, it is not clear how to compute the action of $\mathrm{SL}(2, \mathbb{R})$ on the Hilbert space over $\mathrm{U}(N)/\mathrm{U}(N-1)$. We now introduce another Hilbert space also isomorphic to $\mathbb{S}(V)$ in which the action of $\mathrm{SL}(2, \mathbb{R})$ is readily seen, namely a Hilbert space with Gaussian measure.

Let e_i be an orthonormal basis in V, which then generates a basis $e_{i_1} \otimes \ldots \otimes e_{i_n}$ in the symmetric n-fold tensor product space. Associated with V is its dual V^*, consisting of linear functionals, which can be associated with components v_i of elements in V relative to the basis $\{e_i\}$. Then the natural basis in the n-fold tensor product space corresponds, in the space of functions over V^*, to the polynomials $v_{i_1} \ldots v_{i_n}$; more generally, arbitrary elements in n particle subspaces of $\mathbb{S}(V)$ are associated with degree n polynomials over v_i.

These polynomials are elements of a Hilbert space with Gaussian measure, $L^2(V^*, \mu_G)$, in which the Gaussian measure $\mathrm{d}\mu_G$ satisfies

$$\begin{aligned} &\int_{V^*} \mathrm{d}\mu_G = 1 \\ &\int_{V^*} \mathrm{d}\mu_G \exp[-\mathrm{i}\,\mathrm{Re}(v, w)] = \exp(\|w\|^2/4) \qquad v, w \in V. \end{aligned} \tag{3}$$

It will be given more concretely in the following paragraph. Through the association between basis elements in $\mathbb{S}(V)$ and polynomials in $L^2(V^*, \mu_G)$, it is not hard to show that $\mathbb{S}(V)$ and $L^2(V^*, \mu_G)$ are isomorphic (see, for example, Hida 1980, ch 5).

Since the main groups of interest in this paper are the SO(N) groups, we now restrict our attention to the Hilbert space with Gaussian measure relevant to these groups. The fundamental representation of SO(N) is N dimensional and can be realised on a real vector space of N dimensions, with a dual space $\mathbb{R}^N$. Then the Gaussian measure on the dual space $\mathbb{R}^N$ becomes

$$\int \mathrm{d}\mu_G = \frac{1}{\pi^{N/2}} \int_{\mathbb{R}^N} \mathrm{d}x_1 \ldots \mathrm{d}x_N \exp(-\Sigma x_i^2) \tag{4}$$

and it is easily checked that

$$\int \mathrm{d}\mu_G = \exp[-\mathrm{i}\,\mathrm{Re}(v, w)] = \int_{\mathbb{R}^N} \pi \mathrm{d}x_j \exp(-\Sigma x_j^2) \exp(-\mathrm{i}\Sigma x_j y_j)$$
$$= \exp(-\Sigma y_j^2/4)$$
$$= \exp(-\|y\|^2/4) \qquad y \in V. \tag{5}$$

Let $x \equiv x_1 \ldots x_N$ and $\mathrm{d}\mu_G = \pi\, \mathrm{d}x_j \exp(-\Sigma x_i^2) \equiv \mathrm{d}x \exp(-x^2)$. Then elements of $f \in L^2(\mathbb{R}^N, \mu_G)$ satisfy

$$\|f\|^2 = \int \mathrm{d}\mu_G |f(x)|^2 < \infty$$

and it is seen that polynomials are elements of $L^2(V^*, \mu_G)$; in particular, products of Hermite polynomials $h_{n_1}(x_1) \ldots h_{n_N}(x_N)$ form an orthogonal basis in $L^2(V^*, \mu_G)$, so we define

$$|n_1 \ldots n_N\rangle \equiv |\{n_i\}\rangle = \frac{1}{\sqrt{2^n}} \prod_{i=1}^{N} (n_i!)^{-1/2} h_{n_i}(x_i) \tag{6}$$

and obtain

$$\langle n_1 \ldots n_N | n_1' \ldots n_N' \rangle = \langle \{n_i\} | \{n_i'\} \rangle$$
$$= \frac{1}{(2^n 2^{n'})^{1/2}} \int \mathrm{d}\mu_G \prod_{i=1}^{N} (n_i!\, n_i'!)^{-1/2} h_{n_i}(x_i) h_{n_i'}(x_i)$$
$$= \prod_{i=1}^{N} \delta_{n_i n_i'}. \tag{7}$$

Here $\Sigma_{i=1}^N n_i = n$, $\Sigma_{i=1}^N n_i' = n'$.

The action of $R \in \mathrm{SO}(N)$ on $L^2(\mathbb{R}^N, \mu_G)$ is inherited from the action on $\mathbb{S}(V)$:

$$(\Gamma_R f)(x) = f(R^{-1}x) \qquad f \in L^2(\mathbb{R}^N, \mu_G). \tag{8}$$

We want to find a group with representation operators that act on $L^2(\mathbb{R}^N, \mu_G)$ and commute with Γ_R; if the group action is unitary the scattering operator can be chosen as one of the representation operators, and hence is related to some group element.

Now as pointed out in the beginning of this section, the multiplicity of $\mathrm{SO}(N)$ representations in $\mathbb{S}(V)$ is labelled by n, the particle number. Therefore, the only action of a group commuting with Γ_R can be to change n. In the next section we will show that the number operator is related to an element of the Lie algebra of $\mathrm{SL}(2, \mathbb{R})$ (or $\mathrm{SU}(1, 1)$), and the other operators in the Lie algebra raise or lower the particle number. To prepare for this analysis, we conclude this section by introducing the usual position and momentum operators of non-relativistic quantum mechanics acting on $L^2(\mathbb{R}^N)$ and find the map that sends these operators from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N, \mu_G)$.

Letting $\phi \in L^2(\mathbb{R}^N)$ and defining the map Λ from $L^2(\mathbb{R}^N)$ by

$$(\Lambda\phi)(x) = \pi^{N/4} \exp(\tfrac{1}{2}\Sigma x_i^2)\phi(x) \qquad \begin{array}{l} \Lambda\phi \in L^2(\mathbb{R}^N, \mu_G) \\ \phi \in L^2(\mathbb{R}^N) \end{array} \tag{9}$$

it is easily seen that Λ is unitary ($\|\Lambda\phi\| = \|\phi\|$).

Let q_m and p_m be position and momentum operators satisfying

$$\begin{aligned}(q_m\phi)(x) &= x_m\phi(x)\\ (p_m\phi)(x) &= \frac{1}{\mathrm{i}}\frac{\partial\phi}{\partial x_m} \qquad m=1,\ldots,N.\end{aligned} \tag{10}$$

Then the mapped operators are

$$\begin{aligned}(\Lambda q_m\Lambda^{-1}f)(x) &= x_m f(x)\\ (\Lambda p_m\Lambda^{-1}f)(x) &= \frac{1}{\mathrm{i}}\left(\frac{\partial}{\partial x_m}-x_m\right)f(x)\end{aligned} \tag{11}$$

and are self-adjoint in $L^2(\mathbb{R}^N,\mu_G)$.

Now define

$$H_0=\tfrac{1}{2}\sum_{m=1}^{N}(p_mp_m+q_mq_m) \tag{12a}$$

the N-dimensional harmonic oscillator Hamiltonian, along with

$$H_1=-\tfrac{1}{2}\sum_{m=1}^{N}(p_mp_m-q_mq_m) \tag{12b}$$

$$\begin{aligned}H_2&=-\tfrac{1}{2}\sum_{m=1}^{N}(p_mq_m+q_mp_m)\\ &=\frac{\mathrm{i}N}{2}-\sum_{m=1}^{N}q_mp_m\end{aligned} \tag{12c}$$

where use has been made of the commutator relationship $[q_m,p_m]=\mathrm{i}\delta_{mn}$. The operators H_0, H_1, H_2 form a Lie algebra, the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$.

$$[H_1,H_2]=-2\mathrm{i}H_0 \qquad [H_2,H_0]=2\mathrm{i}H_1 \qquad [H_0,H_1]=2\mathrm{i}H_2. \tag{13}$$

Further, the Casimir invariant

$$C\equiv H_1^2+H_2^2-H_0^2 \tag{14}$$

is not a multiple of the identity, indicating that the $\mathrm{SL}(2,\mathbb{R})$ representation on $L^2(\mathbb{R}^N,\mu_G)$ is reducible. However, all three operators are rotationally invariant, so it is clear that the action of $\mathrm{SL}(2,\mathbb{R})$ commutes with $\mathrm{SO}(N)$.

3. SL(2, $\mathbb{R}$) on $L^2(\mathbb{R}^N, \mu_G)$

The group $\mathrm{SL}(2,\mathbb{R})$ consists of all real 2×2 determinant 1 matrices. A general element can be written as

$$a=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \qquad |a|=1.$$

A basis for the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ can be chosen as

$$\begin{aligned}x_0&=\begin{pmatrix}0&-1\\1&0\end{pmatrix} &\qquad \mathrm{e}^{x_0\theta}&=\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}\\ x_1&=\begin{pmatrix}1&0\\0&-1\end{pmatrix} &\qquad \mathrm{e}^{x_1t}&=\begin{pmatrix}\mathrm{e}^{t}&0\\0&\mathrm{e}^{-t}\end{pmatrix}\\ x_2&=\begin{pmatrix}0&1\\1&0\end{pmatrix} &\qquad \mathrm{e}^{x_2t}&=\begin{pmatrix}\cosh t&\sinh t\\ \sinh t&\cosh t\end{pmatrix}.\end{aligned} \tag{15}$$

Corresponding to each of these Lie algebra elements are self-adjoint operators $\rho(x_i)$ which form a representation of the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ on $L^2(\mathbb{R}^N,\mu_G)$. These operators are obtained via the unitary map Λ carrying elements $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N,\mu_G)$:

$$\begin{aligned}\rho(x_0)&=\Lambda H_0\Lambda^{-1}=x\cdot\nabla-\tfrac{1}{2}\nabla^2+\tfrac{1}{2}N\\ \rho(x_1)&=\Lambda H_1\Lambda^{-1}=\tfrac{1}{2}\nabla^2+x\cdot x-x\cdot\nabla-\tfrac{1}{2}N\\ \rho(x_2)&=\Lambda H_2\Lambda^{-1}=\mathrm{i}(x\cdot\nabla-x\cdot x)+\tfrac{1}{2}\mathrm{i}N.\end{aligned}\tag{16}$$

Further the eigenfunctions of $\rho(x_0)$ are the Hermite polynomial basis elements of $L^2(\mathbb{R}^N,\mu_G)$ defined in the previous section (equation (6)):

$$\begin{aligned}\rho(x_0)|\{n_i\}\rangle&=\tfrac{1}{2}(2x\cdot\nabla-\nabla^2+\tfrac{1}{2}N)\prod_{j=1}^{N}c_jh_{n_j}(x_i)\\ &=\tfrac{1}{2}\sum_{m=1}^{N}(2x_mh'_{n_m}(x_m)-h''_{n_m}(x_m))\prod_{j\neq m}c_jh_{n_j}(x_i)+\tfrac{1}{2}N|\{n_i\}\rangle\\ &=\tfrac{1}{2}\sum_{m=1}^{N}2_{n_m}h_{n_m}(x_m)\prod_{j\neq m}c_jh_{n_j}(x_i)+\tfrac{1}{2}N|\{n_i\}\rangle\\ &=\left(\sum n_m+\tfrac{1}{2}N\right)|\{n_i\}\rangle.\end{aligned}\tag{17}$$

Here use has been made of the Hermite differential equation, $h''_n-2xh'_n+2nh_n=0$; c_j are the normalisation constants given in equation (6). $\rho(x_0)$ thus has the harmonic oscillator spectrum, $n+N/2$, $n=0,1,2,\ldots$, as expected. Also, the Lie algebra element x_0 has been associated with H_0, because its spectrum is discrete, in contrast to the operators $\rho(x_1)$ and $\rho(x_2)$ which have continuous spectra (see Sally 1967, Lange 1975). The dimension factor N appears in each of the $\rho(x_i)$ and is needed to preserve the Lie algebra structure of $\mathrm{SL}(2,\mathbb{R})$.

Since $\mathrm{SL}(2,\mathbb{R})$ commutes with the action of $\mathrm{SO}(N)$ on $L^2(\mathbb{R}^N,\mu_G)$, but is reducible on this space, we must find a space on which the action of $\mathrm{SL}(2,\mathbb{R})$ is irreducible. Now the H_i operators, equation (12), are rotationally invariant, so by passing from cartesian to polar coordinates, and then transforming away the angular coordinates, a representation for $\mathrm{SL}(2,\mathbb{R})$ Lie algebra basis elements in terms of a radial variable only can be found. These transformations are carried out in the appendix. The transformation from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^+,V_l)$ is given by equation (A4) and since the Casimir operator in these radial variables (denoted $\hat{H}_i$) is now a multiple of the identity (equation (A7)ff), it follows that $L^2(\mathbb{R}^+,V_l)$ carries an irreducible representation of the $\mathrm{SL}(2,\mathbb{R})$ Lie algebra.

The irreducible representation carried by $L^2(\mathbb{R}^+,V_l)$ comes from the discrete positive series of representations of $\mathrm{SL}(2,\mathbb{R})$. To see this define $\hat{H}_\pm=\hat{H}_1\pm\mathrm{i}\hat{H}_2$, with commutation relations

$$[\hat{H}_\pm,\hat{H}_0]=\mp2\hat{H}_\pm.\tag{18}$$

Since $\hat{H}_0$ has a spectrum inherited from the harmonic oscillator Hamiltonian, namely $n+N/2$, $n=0,1,2,\ldots$, it follows that $\hat{H}_\pm$ raise or lower n by 2. However in § 2 (equation (2)ff) we showed that n can be written as $n=l$, $l+2$, $l+4,\ldots$, which is precisely the discrete positive series of representations of $\mathrm{SL}(2,\mathbb{R})$ in a basis in which $\rho(x_0)$ is diagonal (i.e. the generator of the compact subgroup of $\mathrm{SL}(2,\mathbb{R})$ is diagonal).

Once the connection with the discrete series has been established, it is a straightforward matter to write the matrix elements of the scattering operator in a partial-wave basis—that is, a basis in which the SO(N) irreducible labels are diagonal. This space is obtained from $L^2(\mathbb{R}^N, \mu_G)$ by transforming away the angular variables, as was done in going from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^+, V_l)$. The unitary transformation from $L^2(\mathbb{R}^N, \mu_G)$ to the partial-wave space $L^2(\mathbb{R}^+, \mu_G, V_l)$ is also given in the appendix (equation (A8)). As was the case with $L^2(\mathbb{R}^+, V_l)$, the Casimir operator on $L^2(\mathbb{R}^+, \mu_G, V_l)$ is a multiple of the identity. The matrix element for a transition from a partial-wave state labelled by l with n' particles to a state with n particles is given by

$$\langle nl\{k\}|S|n'l\{k\}\rangle = \langle nl\{k\}|U_g|n'l\{k\}\rangle \qquad g \in \mathrm{SL}(2, \mathbb{R}) \tag{19}$$

where U_g is a unitary representation operator of the discrete positive series labelled by l, given by Kashiwara and Vergne (1978). Thus l labels the irreducible representation of SL(2, $\mathbb{R}$) and SO(N). $\{k\}$ denotes the set of other basis labels needed to specify a state in the irreducible representation space V_l of SO(N) (Vilenkin 1968). For isospin discussed in the next section $\{k\}$ is the third component of isospin while l is the isospin label itself.

The basis states of the matrix element (19) can be thought of as elements of $L^2(\mathbb{R}^+, \mu_G, V_l)$; however, any other space unitarily equivalent is also suitable for computing the matrix element. Perhaps the most transparent space is the space in which the operator $\rho(x_0)$ is diagonal. The group is then not SL(2, $\mathbb{R}$), but a conjugate group SU(1, 1), in which the compact group element can be written

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

The space on which the compact group elements are represented by diagonal operators is the space of functions holomorphic on the unit disc (Lange 1975). Then the eigenfunctions of $\rho(x_0)$ are simply z^n. The actual matrix elements of SU(1, 1) are given in Vilenkin (1968, p 311ff), and will not be discussed further here.

Matrix elements of the scattering operator in a particle number basis, such as the Hermite polynomial basis given in equation (6), can be obtained by transforming from a $|l|\{k\}\rangle$ basis to an $|\{n_i\}\rangle$ basis with Clebsch-Gordan coefficients of the SO(N) group. A simple method for doing these calculations is given in a succeeding paper in this series.

4. An example—isospin and pion triplets

The simplest example that illustrates the ideas presented in the previous sections makes use of isospin symmetry, in which the internal symmetry group is SU(2) and the representation space V containing a triplet of pions comes from the $I = 1$ representation of SU(2). However, this representation is also the fundamental representation of SO(3), and so fits into the general framework discussed in the previous sections.

The many-particle Fock space of the pions, $\mathbb{S}(V)$, is, as discussed in § 2, isomorphic to $L^2(\mathbb{R}^3, \mu_G)$. A basis in $L^2(\mathbb{R}^3, \mu_G)$ is given by $|n_+, n_0, n_-\rangle$, the number of π^+, π^0 and π^- pions, respectively. The partial-wave amplitude for the transition for n' to n pions in the isospin I state is given by

$$\begin{aligned}\langle n, I, I_3|S|n', I, I_3\rangle &= \langle n, I, I_3|U_g|n', I, I_3\rangle \\ &= \exp\{-i[(n+\tfrac{1}{2}N)\theta - (n'+\tfrac{1}{2}N)\theta']\}\langle n, I, I_3|U_t|n', I, I_3\rangle \\ &= \exp\{-i[(n+\tfrac{1}{2}N)\theta - (n'+\tfrac{1}{2}N)\theta']\}P^I_{nn'}(t) \end{aligned} \tag{20}$$

where U_t is the unitary operator corresponding to the element

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

of SU(1, 1), θ and θ' are parameters of the compact subgroup of SU(1, 1) and $P^{I}_{nn'}(t)$ is the SU(1, 1) matrix element discussed by Vilenkin (1968, p 309ff). The decomposition of $g \in \mathrm{SU}(1, 1)$ used in (20) corresponds to the Euler angle decomposition for SU(2).

Since the dependence of the matrix element on the angles θ, θ' is given explicitly, the partial-wave isospin amplitude is a function of the remaining variable, $t \in \mathbb{R}^+$.

There are several ways that one might fix these parameters in the absence of an underlying dynamical theory. Experimental data or other models might indicate that multiparticle resonances dominate certain channels. In our isospin example, suppose that the $2 \to 4$ reaction proceeds primarily through the $I = 2$ channel, with the $I = 0$ channel suppressed. Then the matrix element $P^{I=0}_{4,2}(t)$ should be small, which restricts the values of t to be near the zeros of $P^{I=0}_{4,2}(t)$. In this way phenomenological input can be used to help fix the value of t and once t has been fixed the partial-wave amplitudes are completely determined.

Finally, to compute the $n' \to n$ matrix element, in which pions $\pi_1 + \ldots + \pi_{n'}$ react to produce pions $\pi_1 + \ldots + \pi_n$, we write

$$\begin{aligned} \langle \pi_1 \ldots \pi_n | S | \pi_{1'} \ldots \pi_{n'} \rangle \\ &= \sum_{I, I_3} \langle \pi_1 \ldots \pi_n | I, I_3 \rangle \langle nI, I_3 | S | n'I, I_3 \rangle \langle I, I_3 | \pi_{1'} \ldots \pi_{n'} \rangle \\ &= \sum_{I, I_3} \langle \pi_1 \ldots \pi_n | I, I_3 \rangle P^{I}_{nn'}(t) \langle I, I_3 | \pi_{1'} \ldots \pi_{n'} \rangle \end{aligned} \tag{21}$$

where $\langle \pi_1 \ldots \pi_n | I, I_3 \rangle$ is a SO(3) Clebsch–Gordan coefficient in which n $I = 1$ particles with third component of isospin $\pi_1 \ldots \pi_n$ are symmetrically coupled together to produce an isospin I and component I_3 state. These coefficients are discussed by Klink (1983).

5. Conclusion

We have shown that for SO(N) internal symmetry groups, in which the irreducible N-dimensional representation space of SO(N) generates a symmetric Fock space, that a general unitary invariant scattering operator is related to the positive discrete series of representations of SL(2, $\mathbb{R}$). In particular, unitary invariant partial-wave amplitudes of SO(N) internal symmetry groups can be written as SL(2, $\mathbb{R}$) matrix elements, in which the irreducible representation of SL(2, $\mathbb{R}$) is given by an irreducible representation of SO(N), while the basis labels are the number of particles in the initial and final states.

For example, isospin symmetry generated by SU(2), in which the $I = 1$ representation of SU(2) contains a triplet of pions, can be thought of as an SO(3) internal symmetry with the three-dimensional representation generating the Fock space. Then isospin partial-wave amplitudes are given as matrix elements of SL(2, $\mathbb{R}$) (which are well known special functions) in which the discrete series of representations of SL(2, $\mathbb{R}$) are labelled by isospin and the basis labels are the number of pions in the initial and final states. Such partial-wave amplitudes are not, of course, the most general isospin partial-wave amplitudes, as they come from a finite parameter subgroup $G = SL(2, \mathbb{R})$ of the full infinite parameter group of transformations that commute with SU(2).

The notion of a group G whose action on a Hilbert space commutes with the action of a given compact group K is closely related to work by Moshinsky, Quesne and co-workers (Moshinsky and Quesne 1970, 1971, Deenen and Quesne 1982, Moshinsky 1984, Quesne 1985), who call such pairs of groups complementary groups, and Howe (1985), who calls such pairs of groups 'dual pairs'. Their work makes use of the fact that $SO(N)$ and $SL(2, \mathbb{R})$ are complementary or dual, and that a given $SO(N)$ irreducible representation carries a unique representation of the discrete series of $SL(2, \mathbb{R})$. Complementarity or duality ideas have also been used to analyse the branching rules and matrix elements of the infinite-dimensional unitary representations of the real symplectic groups through the association with the finite-dimensional irreducible representations of $SO(N)$ (Rowe *et al* 1985).

Howe has listed various sets of dual pairs of groups. This list unfortunately does not include the compact internal symmetry groups of most physical interest. For example, a multiplet of eight bosons is generated by the eight-dimensional representation of $SU(3)_{\text{flavour}}$. The operators of SU(3) acting on this eight-dimensional representation form a subgroup of SO(8). Therefore, the group dual or complementary to this subgroup of SO(8) will contain $SL(2, \mathbb{R})$, but also other elements. How to find such a group will be discussed in another paper.

Of most physical interest is finding groups dual or complementary to spacetime groups such as the Poincaré group. In this case the underlying Fock space is generated by an infinite-dimensional irreducible representation and so does not fit in any natural way to dual or complementary groups previously given. A way of finding elements of the dual pairs by using group contraction ideas is given in Klink (1987).

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Appendix

We want transformations between $L^2(\mathbb{R}^N)$ and $L^2(\mathbb{R}^+, V_l)$, $L^2(\mathbb{R}^N, \mu_G)$ and $L^2(\mathbb{R}^+, \mu_G, V_l)$, and finally between $L^2(\mathbb{R}^+, V_l)$ and $L^2(\mathbb{R}^+, \mu_G, V_l)$, because on these latter spaces $SL(2, \mathbb{R})$ acts irreducibly.

To obtain the map between $L^2(\mathbb{R}^N)$ and $L^2(\mathbb{R}^+, V_l)$, we make a change from cartesian to spherical coordinates, $x = r\Omega$, where Ω are angular coordinates, given, for example, by Vilenkin (1968, p 435 or 489). Ω denotes any choice of spherical coordinates. The measure then transforms as

$$\prod_{i=1}^{N} \mathrm{d}x_i = r^{N-1}\, \mathrm{d}r\, \mathrm{d}\Omega \tag{A1}$$

where $\mathrm{d}\Omega$ is normalised so that

$$\int_{SO(N)/SO(N-1)} \mathrm{d}\Omega = 1.$$

Let $Y_{l\{k\}}(\Omega)$ be generalised spherical harmonics (Gegenbauer polynomials) satisfying

$$(Y_{l\{k\}}, Y_{l'\{k'\}}) = \int_{\mathrm{SO}(N)/\mathrm{SO}(N-1)} \mathrm{d}\Omega\, Y^*_{l\{k\}}(\Omega) Y^*_{l'\{k'\}}(\Omega)$$
$$= \delta_{ll'}\delta_{\{k\},\{k'\}}. \tag{A2}$$

Here l is the irreducible representation label of SO(N) (it is $(l0\ldots0)$ in Gel'fand notation, with $N/2-1$ zeros if N is even and $\frac{1}{2}(N-1)-1$ zeros if N is odd). $\{k\}$ stands for the collective indices needed to further specify the generalised spherical harmonics.

Let $L^2(\mathbb{R}^+, V_l)$ be the Hilbert space in which elements $\hat{\phi} \in L^2(\mathbb{R}^+, V_l)$ have norm

$$\|\hat{\phi}\|^2 = \sum_l \sum_{\{k\}} \int_{\mathbb{R}^+} r^{N-1}\, \mathrm{d}r |\hat{\phi}(r, l\{k\})|^2 < \infty \tag{A3}$$

and define the unitary map from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^+, V_l)$ by

$$(\Lambda\phi)(r, l, \{k\}) = \int_{\mathbb{R}^+} r^{N-1}\, \mathrm{d}r$$
$$\times \int_{\mathrm{SO}(N)/\mathrm{SO}(N-1)} \mathrm{d}\Omega\; Y^*_{l\{k\}}(\Omega)\phi(r\Omega) \qquad \phi \in L^2(\mathbb{R}^N). \tag{A4}$$

The inverse map is given by

$$(\Lambda^{-1}\hat{\phi})(x) = \sum_l \sum_{\{k\}} Y_{l\{k\}}(\Omega)\hat{\phi}(r, l, \{k\}). \tag{A5}$$

In $L^2(\mathbb{R}^+, V_l)$, the various operators become

$$\begin{aligned}
\Lambda\nabla^2\Lambda^{-1} &= \frac{1}{r^{N-1}}\frac{\partial}{\partial r}r^{N-1}\frac{\partial}{\partial r} - \frac{l(l+N-2)}{r^2}\\
&= \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} - \frac{l(l+N-2)}{r^2}\\
\Lambda(x\cdot\nabla)\Lambda^{-1} &= r\partial/\partial r\\
\Lambda(x\cdot x)\Lambda^{-1} &= r^2
\end{aligned} \tag{A6}$$

and the Lie algebra elements are

$$\begin{aligned}
\hat{H}_0 &= \Lambda H_0\Lambda^{-1} = \frac{1}{2}\left(-\frac{\partial^2}{\partial r^2} - \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{l(l+N-2)}{r^2} + r^2\right)\\
\hat{H}_1 &= \Lambda H_1\Lambda^{-1} = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} - \frac{l(l+N-2)}{r^2} + r^2\right)\\
\hat{H}_2 &= \Lambda H_2\Lambda^{-1} = \frac{\mathrm{i}N}{2} + \mathrm{i}r\frac{\partial}{\partial r}.
\end{aligned} \tag{A7}$$

Most significantly, the Casimir operator $\hat{C} \equiv \hat{H}_1^2 + \hat{H}_2^2 - \hat{H}_0^2 = 2N - (N/2)^2 - l(l+N-2)$, is a multiple of the identity, so that on $L^2(\mathbb{R}^+, V_l)$, SL(2, $\mathbb{R}$) acts irreducibly.

Similarly, a unitary map Λ_G from $L^2(\mathbb{R}^N, \mu_G)$ to the 'partial-wave' space $L^2(\mathbb{R}^+, \mu_G, V_l)$ can be defined as

$$(\Lambda_G f)(r, l, \{k\}) = \frac{1}{\pi^{N/2}} \int_{\mathbb{R}^+} r^{N-1} \mathrm{e}^{-r^2} \,\mathrm{d}r \times \int_{\mathrm{SO}(N)/\mathrm{SO}(N-1)} \mathrm{d}\Omega \, Y^*_{l\{k\}}(\Omega) \phi(r\Omega) \qquad f \in L^2(\mathbb{R}^N, \mu_G). \tag{A8}$$

Here $\hat{f} \in L^2(\mathbb{R}^+, \mu_G, V_l)$ means

$$\|\hat{f}\|^2 = \frac{1}{\pi^{N/2}} \sum_{l,\{k\}} \int_{\mathbb{R}^+} r^{N-1} \mathrm{e}^{-r^2} \,\mathrm{d}r |\hat{f}(r, l\{k\})|^2 < \infty. \tag{A9}$$

Finally, the map between $L^2(\mathbb{R}^+, V_l)$ and $L^2(\mathbb{R}^+, \mu_G, V_l)$ is given by

$$(\Lambda \hat{\phi})(r, l, \{k\}) = \pi^{N/4} \mathrm{e}^{r^2/2} \hat{\phi}(r, l, \{k\}) \qquad \hat{\phi} \in L^2(\mathbb{R}^+, V_l) \tag{A10}$$

and we check that this is unitary, i.e.

$$\begin{aligned} \|\Lambda \hat{\phi}\|^2 &= \frac{1}{\pi^{N/2}} \sum_l \sum_{\{k\}} \int_{\mathbb{R}^+} r^{N-1} \mathrm{e}^{-r^2} \,\mathrm{d}r |\pi^{N/4} \mathrm{e}^{r^2/2} \hat{\phi}(r, l\{k\})|^2 \\ &= \|\hat{\phi}\|^2. \end{aligned} \tag{A11}$$

The operators defining the Lie algebra elements become

$$\begin{aligned} \Lambda \nabla^2 \Lambda^{-1} &= \frac{\partial^2}{\partial r^2} + \left(\frac{N-1}{r} - 2r \right) \frac{\partial}{\partial r} + (r^2 - N) - \frac{l(l+N-2)}{r^2} \\ \Lambda r^2 \Lambda^{-1} &= r^2 \\ \Lambda r \frac{\partial}{\partial r} \Lambda^{-1} &= r \frac{\partial}{\partial r} - r^2 \end{aligned} \tag{A12}$$

with Lie algebra elements

$$\begin{aligned} \hat{\rho}(x_0) &= \frac{1}{2} \left[-\frac{\partial^2}{\partial r^2} - \left(\frac{N-1}{r} - 2r \right) \frac{\partial}{\partial r} - (r^2 - N) + \frac{l(l+N-2)}{r} + r^2 \right] \\ \hat{\rho}(x_1) &= \frac{1}{2} \left[\frac{\partial^2}{\partial r^2} + \left(\frac{N-1}{r} - 2r \right) \frac{\partial}{\partial r} + (r^2 - N) - \frac{l(l+N-2)}{r^2} + r^2 \right] \\ \hat{\rho}(x_2) &= \frac{\mathrm{i}N}{2} + \mathrm{i} \left(r \frac{\partial}{\partial r} - r^2 \right) \end{aligned}$$

and a Casimir operator which is a multiple of the identity.

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